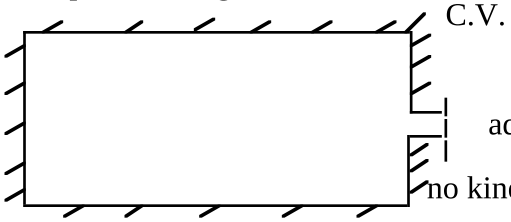
4.2.1 Isentropic Discharge Process



adiabatic, rigid control volume,
equilibrium inside the tank,
no kinetic and potential energy changes.

$$\dot{Q} - \dot{W} + \sum_{in} \dot{m}_i h_i - \sum_{out} \dot{m}_e h_e = (\dot{E})_{c.v.}$$

$$\dot{Q} = \dot{W} = \dot{m}_i = 0$$

$$\left(\frac{dE}{dt}\right)_{c.v.} = - \dot{m}_e h_e$$

$$\frac{d}{dt}(mu)_{c.v.} = \left(\frac{dm}{dt}\right)_{cv} \bullet h_e; \int \frac{d}{dt}(mu)_{cv} = \int \left(\frac{dm}{dt}\right) he$$

where u is the internal energy and h is the enthalpy

$$d(mu)_{c.v.} = h\, dm \text{ or}$$

$$h\, dm = m\, du + u\, dm$$

$$(h - u)\, dm = m\, du$$

$$\boxed{\frac{du}{h - u} = \frac{dm}{m}}$$

$$\frac{dm}{m} = -\frac{V\, dv}{V\, v} = -\frac{dv}{v} \Rightarrow \left(\frac{dm}{m} = \frac{d(V/v)}{(V/v)} = \frac{V}{V}\frac{d\,1/v}{Vv} = -v \bullet \frac{1}{v^2} \bullet dv = -\frac{dv}{v}\right)$$

h - u = pv; where V is the total volume and v is the specific volume

$$\frac{du}{pv} = - \frac{dv}{v}$$

or $\boxed{du + p\, dv = 0}$

It is also known (from Gibbs Equation) that T ds = du + p dv, s is the entropy; thus,

$$\boxed{ds = 0}$$

The isentropic process follows

$$p_1 v_1^k = p_2 v_2^k \quad \text{where k is the specific heat ratio} \equiv \frac{Cp}{Cv}$$

$$\frac{p_2}{p_1} = \left(\frac{v_1}{v_2}\right)^k = \left(\frac{R\ T_1/p_1}{R\ T_2/p_2}\right)^k$$

$$\frac{p_2}{p_1} = \left(\frac{T_1}{T_2}\right)^k \left(\frac{p_2}{p_1}\right)^k, \text{ or}$$

$$\boxed{\left(\frac{T_2}{T_1}\right) = \left(\frac{p_2}{p_1}\right)^{(k-1)/k}}$$

$$\frac{m_2}{m_1} = \left(\frac{V/v_2}{V/v_1}\right) = \left(\frac{v_1}{v_2}\right) = \left(\frac{p_2}{p_1}\right)^{1/k}$$

$$\boxed{\frac{m_2}{m_1} = \left(\frac{p_2}{p_1}\right)^k}$$

<u>Example</u> Discharge process of Experiment No. 3

It was shown earlier that the discharge process is an isentropic process; conversely,

$$\frac{P_t}{P_0} = \left(\frac{v_0}{v_t}\right)^k = \left(\frac{\rho_t}{\rho_0}\right)^k$$

where "t" and "0" denote the tank pressure at time t and initial pressure. Recall the IG EOS:

$$P_t = \rho_t RT_t$$

Thus, one may relate the system mass to its thermodynamic state.

$$\frac{dm}{dt} = - \dot{m}_e = V \frac{d\rho_t}{dt} = V \frac{d\,(P_t/RT_t)}{dt}$$

where V is the volume of the tank. Substituting the isentropic relationship into the above equation, one obtains

$$\dot{m}_e = - V \frac{d\left(\frac{p_t}{p_0}\right)^{1/k} \bullet \rho_0}{dt}$$

$$\dot{m}_e = - V\rho_0 \left(\frac{p_t}{p_0}\right)^{1/k - 1} \frac{1}{k} \frac{d\,(p_t/p_0)}{dt}$$

$$\dot{m}_e = - \frac{V\rho_0}{k} \left(\frac{p_t}{p_0}\right)^{(1 - k) / k} \frac{d(p_t/p_0)}{dt} \qquad \text{(A)}$$

Let's define the discharge coefficient:

$$\dot{m}_e = C_D \, \dot{m}_{e,\,i}$$

where c_D is the discharge coefficient and $\dot{m}_{e,\,i}$ is the ideal mass flow rate. For a converging nozzle, the ideal flow rate is the choked mass flow if the tank pressure is higher than the critical pressure, P_*. The critical pressure, P_*, can be determined from

- $$\frac{P_a}{P_*} = \left(\frac{2}{k + 1}\right)^{k / (k - 1)}$$

  As an example, P_a = 14.7 psia, k = 1.4 for air, P_* = 27.8 psia (P_a / P_* = 0.5283 ; P_* = 1.8929 P_a)

The choked flow condition has a mass flow rate

$$\dot{m}_* = \frac{A\,P_t}{\sqrt{R\,T_t}} \sqrt{k} \left(\frac{2}{k + 1}\right)^{(k + 1) / [2(k - 1)]} \qquad \text{(B)}$$

where A is the throat or nozzle opening area. This equation is valid when $P_t > P_*$. When $P_t / P_a < P_* / P_a$, it can be shown that

$$\dot{m} = \frac{A\,P_t}{\sqrt{R\,T_t}} \sqrt{\frac{2k}{k - 1}\left[\left(\frac{P_a}{P_t}\right)^{2 / k} - \left(\frac{P_a}{P_t}\right)^{(k + 1) / k}\right]} \qquad \text{(C)}$$

Substituting the choked flow equation into the mass flow rate expression, i.e., combining Eqs. (B) & (A), one obtains

$$-\frac{V\rho_o}{k}\left(\frac{P_t}{P_o}\right)^{(1-k)/k}\frac{d(P_t/P_o)}{dt} = \dot{m}_e = C_D\frac{A\ P_t}{\sqrt{R\ T_t}}\sqrt{k}\left(\frac{2}{k+1}\right)^{(k+1)/[2(k-1)]}$$

$$\left(\frac{P_t}{P_o}\right)^{(1-3k)/2k}\frac{d(P_t/P_o)}{dt} = -C_D\frac{A\ k\sqrt{kR\ T_o}}{V}\left(\frac{2}{k+1}\right)^{(k+1)/[2(k-1)]}$$

The solution to the above equation with an I.C. of $P_t = P_o$ at $t = 0$ is

<u>Choked Flow</u>

$$-\frac{V\rho_o}{k}\left(\frac{P_t}{P_o}\right)^{(1-k)/k}\frac{d(P_t/P_o)}{dt} = C_D\frac{A\ P_t}{\sqrt{R\ T_t}}\sqrt{k}\left(\frac{2}{k+1}\right)^{(k+1)/[2(k-1)]}$$

$$\left(\frac{P_t}{P_o}\right)^{(1-k)/k}\frac{d(P_t/P_o)}{dt} = -C_D\frac{A\sqrt{k}}{V}\frac{k}{\rho_o}\frac{P_t}{\sqrt{R\ T_t}}\left(\frac{2}{k+1}\right)^{(k+1)/[2(k-1)]}$$

$$= -C_D\frac{A\sqrt{k}}{V}\bullet\frac{P_t}{P_o}\bullet\frac{\rho_o R\ T_o}{\rho_o\sqrt{R\ T_t}}\bullet k\left(\frac{2}{k+1}\right)^{(k+1)/[2(k-1)]}$$

$$= -C_D\frac{Ak\sqrt{k}}{V}\bullet\frac{P_t}{P_o}\bullet\sqrt{R\ T_o}\sqrt{\frac{T_o}{T_t}}\bullet\left(\frac{2}{k+1}\right)^{(k+1)/[2(k-1)]}$$

$$\frac{P_t}{P_o}\bullet\sqrt{\frac{T_o}{T_t}} = \frac{P_t}{P_o}\bullet\left(\frac{P_o}{P_t}\right)^{(k-1)/2k} = \left(\frac{P_t}{P_o}\right)^{(k+1)/2k}$$

$$\left(\frac{P_t}{P_o}\right)^{(1-3k)/2k}\bullet\frac{d(P_t/P_o)}{dt} = -C_D\frac{A\ k\sqrt{kR\ T_o}}{V}\left(\frac{2}{k+1}\right)^{(k+1)/[2(k-1)]}$$

$$\frac{2k}{1-k}\bullet\frac{d(P_t/P_o)^{(1-k)/2k}}{dt} = -C_D\frac{A\ k\sqrt{kR\ T_o}}{V}\left(\frac{2}{k+1}\right)^{(k+1)/[2(k-1)]}$$

Applying the initial condition $t = 0$, $P_t = P_0$, one obtains

$$\left(\frac{P_t}{P_o}\right)^{(1-k)/2k} - 1 = -C_D\frac{A\ \not{k}\sqrt{kR\ T_o}}{V}\bullet\frac{1-k}{2\not{k}}\bullet\left(\frac{2}{k+1}\right)^{(k+1)/[2(k-1)]}\bullet t$$

$$\left(\frac{P_o}{P_t}\right)^{(k-1)/2k} = 1 + C_D\frac{A\sqrt{kR\ T_o}}{V}\bullet\frac{k-1}{2}\bullet\left(\frac{2}{k+1}\right)^{(k+1)/[2(k-1)]}\bullet t$$

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$$\boxed{\frac{P_t}{P_o} = \left[1 + C_D\frac{A}{V}\sqrt{kR\ T_o}\bullet\frac{k-1}{2}\bullet\left(\frac{2}{k+1}\right)^{(k+1)/[2(k-1)]}t\right]^{2k/(1-k)}}$$

Also, one obtains

$$t = \frac{V}{C_D\ A\sqrt{kR\ T_o}} \left(\frac{2}{k-1}\right)\left(\frac{k+1}{2}\right)^{(k+1)/[2(k-1)]} \left[\left(\frac{P_o}{P_t}\right)^{(k-1)/2k} - 1\right]$$

which can be used to obtain the time to reach P*, i.e. t* by setting P_t = P*. For t> t*, the mass flow rate can be obtained by considering Eqns. (A) and (C):

Unchoked Flow

$$\frac{\dfrac{d(P_t/P_a)}{dt}}{\left(\dfrac{P_t}{P_a}\right)^{(k-1)/k}\sqrt{\left(\dfrac{P_t}{P_a}\right)^{(k-1)/k} - 1}} = -\frac{C_D\ Ak\sqrt{kR\ T_o}}{V}\left(\frac{2}{k-1}\right)^{1/2}\left(\frac{P_a}{P_o}\right)^{(k-1)/2k}$$

A closed form solution was obtained by Owczarek (1964) using the following transformation:

$$x = \sqrt{\left(\frac{P_t}{P_a}\right)^{(k-1)/k} - 1}$$

$$t - t_* = \frac{2k}{k-1}\left[\frac{V}{C_D\ A}\left(\frac{k-1}{2}\right)^{1/2}\frac{(P_o/P_a)^{(k-1)/2k}}{k\sqrt{kR\ T_o}}\right]$$

$$\bullet\left[0.492 - \frac{x}{8}(2x^2+5)\sqrt{x^2+1} - \frac{3}{8}\ln\left(x + \sqrt{x^2+1}\right)\right]$$

Reference

Owczarek, J.A., Fundamentals of Gas Dynamics, International Textbook Co., Scranton, Pa, 1964.